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REVERBERATION MATRIX METHOD FOR PROPAGATION OF SOUND IN A MULTILAYERED LIQUID

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This paper presents the reverberation matrix method for wave propagation in a multi-layered liquid. First the local scattering matrix and phase matrix for the reflected and refracted waves are derived at each interface of two layers in terms of local co-ordinates. The local matrices of all layers are then stacked to form global scattering and global phase matrices. The product of these two matrices together with a global permutation matrix gives rise to the reverberation matrix R which represents the propagation of steady state waves through the multi-layered medium. By expanding the matrix $[\mathbf{I} - \mathbf{R}]^{-1}$ into a power series and applying the inverse Fourier transform, we then derive the ray integrals for transient waves generated by a column of point sources and propagating through multi-reflected and refracted paths in the medium. The ray integrals so derived are particularly suitable for numerical calculations by applying the Cagniard method.

1. INTRODUCTION

The propagation of sound in stratified liquid was measured and analyzed as early as 1940s. A complete report of experiments in ocean, observations and mathematical analysis was published after World War II by Ewing *et al.* [1]. In that report, Pekeries proposed a layered liquid model to analyze the experimental data of sound waves in shallow water. The propagation of transient waves is represented by a double integral of the steady state wave function, one with respect to the wave number κ and the other with respect to the frequency ω, and the integration can be carried out by two procedures. One is called the normal-mode method and the other is called the ray method. His theory and analysis formed the basis for much of later investigations, as reviewed by Tolstoy and Clay [2] and Brekhovskikh [3].

In the mean time, the theory and analysis were also applied to layered solid model for the Earth. The dispersion relation for steady state waves in multilayered media was formulated in matrix form by Thomson [4] and Haskell [5], and the

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analysis of the numerical evaluations of the dispersion matrix for various layered models was summarized by Ewing *et al.* [6]. Many other matrix formulations for the dispersion relation were proposed later, and they could all be considered as special cases of general propagator matrices by Gilbert and Backus [7]. For transient wave analysis, Kennett and Kerry [8] developed the reflection matrix formulation for wave propagating from the source to receiver to analyze the seismic waves in a stratified half-space.

In 1980s Lu and Felsen [9] presented a Green matrix method to analyze waves in multilayered media. Their formulation is particularly suitable for the hybrid normal mode-ray analysis, but it is not convenient for applying the Cagniard method to evaluate the transient waves precisely along generalized ray paths. Recently, Howard and Pao [10] developed a reverberation matrix method for the scattering of one-dimensional waves at the structural joints and the multireflections of waves between the joints of a truss or frame. In this article, their method is extended to study the propagation of sound waves in a layered liquid. The matrix formulation was developed originally for analyzing the transient waves in trusses and frames. The method as presented is simple to interpret physically, and can be easily adopted to analyze the transient waves by applying the Cagniard method.

The formulation of the reverberation matrix will be discussed in the next three sections. A set of local co-ordinates is introduced for each layer. The wave potential in each layer is represented and transformed into the spectral domain by applying Fourier transform and Hankel transform (section 2). In the spectral domain, the scattering matrix for transferring arrival waves to departure waves at an interface or a boundary is derived for appropriate boundary and continuity conditions (section 3). A reverberation matrix through the multilayered medium is then formulated from the product of global scattering matrix, phase matrix and permutation matrix (section 4). In section 5, transient waves at multiple receivers are expressed by inverting the Hankel and Fourier transforms. The integrals of inverse transform may be evaluated by the normal-mode method, ray expansion method or hybrid method. Section 6 shows a numerical example, and the final section contains a conclusion.

2. AXISYMMETRIC SOUND WAVES IN A LAYERED LIQUID

Consider a multilayered liquid separated by parallel planes $z = Z^J (J = 0, 1, 2, \ldots, N)$, where the axisymmetric co-ordinate system (r, z) is shown in Figure 1. All sources are assumed to be located at an interface between two layers. If the source is situated at the interior of a layer, then an additional interface passing the layer is added artificially to divide the original layer into two portions of equal material properties. We shall designate the plane interface with $I, J, K, \ldots$ and layers with two capital letters. All physical quantities at the interface $z = Z^J$ will carry the superscript J; those at the layer bounded by two adjacent interfaces $z = Z^J$ and $z = Z^K$ carry two superscripts JK. Thus, the mass density, bulk modulus and sound speed in the layer are denoted by ρ^{JK}, k^{JK}, c^{JK}; the force applied at the interface $z = Z^J$ by the vector $\mathbf{f}^J$.

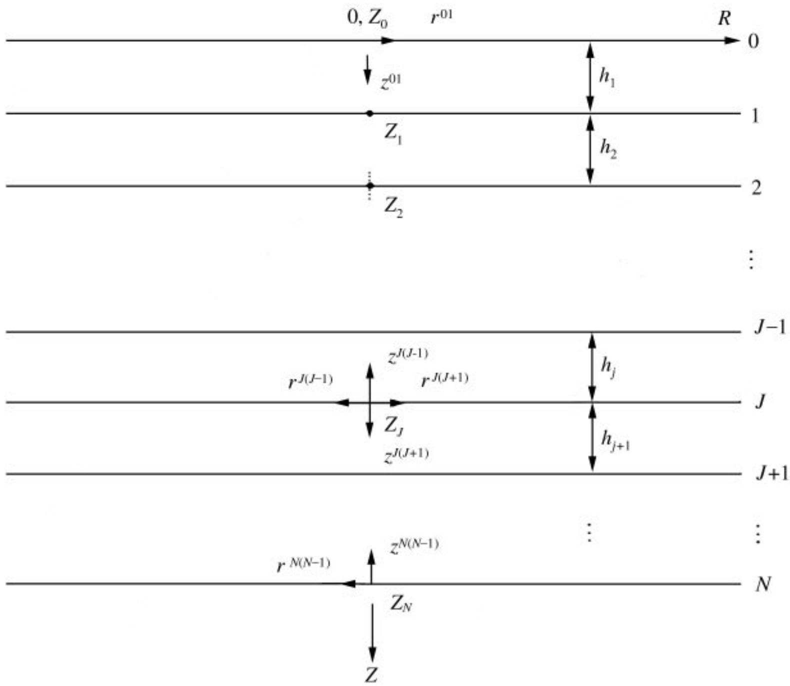

Figure 1. Geometry and co-ordinates in a multilayered medium.

In this section, however, we shall omit all superscripts for wave quantities in each layer.

The sound wave pressure p and the particle velocity along the radial and azimuth co-ordinates, (u, w), are determined respectively from the wave potential $\phi(r, z, t)$ by the relations

$$p = \rho \frac{\partial \phi}{\partial t}, \qquad w = -\frac{\partial \phi}{\partial z}, \qquad u = -\frac{\partial \phi}{\partial r}, \tag{1}$$

The wave function $\phi(r, z, t)$ satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}, \tag{2}$$

where $c = (k/\rho)^{1/2}$.

The Fourier transform of $\phi(r, z, t)$ in time variable t and the inverse Fourier transform of $\bar{\phi}(r, z, \omega)$ are given by

$$\tilde{\phi}(r, z, \omega) = \int_{-\infty}^{\infty} \phi(r, z, t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t, \tag{3}$$

$$\phi(r, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\phi}(r, z, \omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega. \tag{4}$$

The Hankel transform of $\tilde{\phi}(r, z, \omega)$ and the inverse Hankel transform of $\hat{\phi}(\kappa, z, \omega)$ are given by

$$\hat{\phi}(\kappa, z, \omega) = \int_0^\infty \tilde{\phi}(r, z, \omega) J_0(\kappa r) r \, \mathrm{d}r, \tag{5}$$

$$\tilde{\phi}(r, z, \omega) = \int_0^\infty \hat{\phi}(\kappa, z, \omega) J_0(\kappa r) \kappa \, \mathrm{d}\kappa. \tag{6}$$

By applying Fourier transform and Hankel transform, equation (2) is reduced to

$$\frac{\mathrm{d}^2 \hat{\phi}}{\mathrm{d}z^2} + \alpha_j^2 \hat{\phi} = 0, \tag{7}$$

where $\alpha = (\omega^2/c^2 - \kappa^2)^{1/2}$. The solution for the equation can be expressed as

$$\hat{\phi} = \hat{a}\mathrm{e}^{-\mathrm{i}\alpha z} + \hat{d}\mathrm{e}^{\mathrm{i}\alpha z}, \tag{8}$$

where $\hat{a}$ and $\hat{d}$ are unknown coefficients. Furthermore, the twice transformed pressure and vertical velocity are given by

$$\hat{p} = \mathrm{i}\omega\rho\hat{\phi}, \qquad \hat{w} = -\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}z}. \tag{9}$$

3. SCATTERING MATRICES FOR WAVES AT INTERFACES

We restore the superscripts to all physical quantities and introduce a set of local co-ordinates (r^{JK}, z^{JK}) for each layer above $(K = J - 1)$ and below $(K = J + 1)$ the interface J, as shown in Figure 1. Because of symmetry, we have

$$r^{J(J-1)} = r^{J(J+1)} = r \tag{10}$$

and

$$z^{JK} = h^{JK} - z^{KJ}, \tag{11}$$

where $h^{JK} = h^{KJ}$ is the thickness of the layer JK. Within the layer JK, the two superscripts for ϕ, ρ and c are interchangeable. For convenience, the material parameters of medium in the jth layer are also represented by subscripts, such as ρ_j, c_j, … , etc.

3.1. LOCAL SCATTERING MATRIX AT INTERFACE J

From equation (8), the potentials in the two adjacent layers at interface J are expressed respectively by

$$\hat{\phi}^{J(J-1)}(k, z^{J(J-1)}, \omega) = \hat{a}^{J(J-1)} \mathrm{e}^{-\mathrm{i}\alpha_j z^{J(J-1)}} + \hat{d}^{J(J-1)} \mathrm{e}^{\mathrm{i}\alpha_J z^{J(J-1)}},$$

$$\hat{\phi}^{J(J+1)}(k, z^{J(J+1)}, \omega) = \hat{a}^{J(J+1)} \mathrm{e}^{-\mathrm{i}\alpha_{j+1} z^{J(J+1)}} + \hat{d}^{J(J+1)} \mathrm{e}^{\mathrm{i}\alpha_{J+1} z^{J(J+1)}}. \qquad (12)$$

Associated with the time factor $\mathrm{e}^{-\mathrm{i}\omega t}$ in equation (4), the term with unknown amplitude $\hat{d}^{JK}$ represents a wave departing from the interface J and travelling in the positive direction of z^{JK}; and that with $\hat{a}^{JK}$ represents a wave arriving at the interface J and travelling in the negative direction of z^{JK}. $\hat{a}^{JK}$ and $\hat{d}^{JK}$ are unknown functions of κ and ω, which will be determined by the boundary conditions and the continuity conditions at the interfaces.

If an explosive source with time function $f(t)$ is placed at $r = 0$, $z = Z^J$, the origin of two local co-ordinates, the source function may be represented by $(1/2\pi r)\delta(r-0)\delta(z-0^J)f(t)$. The pressure should be continuous at the interface, but the velocity is not. Because the fluid above and below the source moves in opposite directions, the vertical velocity will jump across the interface. We can obtain two continuity conditions at the interface.

$$\hat{p}^{J(J+1)}(\kappa, 0, \omega) - \hat{p}^{J(J-1)}(\kappa, 0, \omega) = 0,$$

$$\hat{w}^{J(J+1)}(\kappa, 0, \omega) + \hat{w}^{J(J-1)}(\kappa, 0, \omega) = g_w^J, \quad J = 1, 2, \ldots, N-1, \qquad (13)$$

where $g_w^J = \bar{f}(\omega)/4\pi[c_{j+1}^{-2} + c_j^{-2}]$. Substituting equations (12) and (9) into the previous equations, we obtain a set of equations for the unknown coefficients $\hat{a}^{J(J-1)}$, $\hat{a}^{J(J-1)}$, $\hat{d}^{J(J-1)}$ and $\hat{d}^{J(J+1)}$, which is expressed in matrix form as follows:

$$A^J \hat{\mathbf{a}}^J + D^J \hat{\mathbf{d}}^J = \hat{\mathbf{g}}^J(\kappa, \omega), \qquad (14)$$

where $\hat{\mathbf{a}}^J$ and $\hat{\mathbf{d}}^J$ are unknown vectors, A^J and D^J are 2×2 matrices, and $\hat{\mathbf{g}}^J$ is the external force vector, i.e.,

$$\hat{\mathbf{a}}^J = (\hat{a}^{J(J-1)}, \hat{a}^{J(J+1)})^{\mathrm{T}}, \qquad \hat{\mathbf{d}}^J = (\hat{d}^{J(J-1)}, \hat{d}^{J(J+1)})^{\mathrm{T}},$$

$$A^J = \begin{bmatrix} -\rho_j & \rho_{j+1} \\ -\mathrm{i}\alpha_j & -\mathrm{i}\alpha_{j+1} \end{bmatrix}, \quad D^J = \begin{bmatrix} -\rho_j & \rho_{j+1} \\ \mathrm{i}\alpha_j & \mathrm{i}\alpha_{j+1} \end{bmatrix},$$

$$\hat{\mathbf{g}}^J(\kappa, \omega) = (g_p^J, g_w^J)^{\mathrm{T}},$$

$$g_p^J = 0, \qquad g_w^J = \frac{\bar{f}(\omega)}{4\pi}[c_{j+1}^{-2} + c_j^{-2}].$$

Obviously, both components of the force vector vanish when no source is at the interface.

Solving $\hat{\mathbf{d}}^J$ in terms of unknown vector $\hat{\mathbf{a}}^J$ and a given source vector $\hat{\mathbf{g}}^J$, we find

$$\hat{\mathbf{d}}^J = S^J \hat{\mathbf{a}}^J + \hat{\mathbf{s}}^J(\kappa, \omega), \quad J = 1, 2, \ldots, N-1, \tag{15}$$

where

$$\begin{aligned} S^J &= -(D^J)^{-1} A^J \\ &= \frac{1}{\Delta^J} \begin{bmatrix} \rho_{j+1}\alpha_j - \rho_j \alpha_{j+1} & 2\rho_{j+1}\alpha_{j+1} \\ 2\rho_j \alpha_j & \rho_j \alpha_{j+1} - \rho_{j+1}\alpha_j \end{bmatrix}, \\ \Delta^J &= \rho_j \alpha_{j+1} + \rho_{j+1}\alpha_j \end{aligned} \tag{16}$$

and

$$\hat{\mathbf{s}}^J(\kappa, \omega) = (D^J)^{-1} \hat{\mathbf{g}}^J(\kappa, \omega). \tag{17}$$

The matrix S^J is called the scattering matrix at the Jth interface, the element of which relates an incident wave (arrival) to the transmitted or reflected wave (departure) in the Jth interface. The $\hat{\mathbf{s}}^J$ is called the source wave vector, which represents the waves emitted by the explosive source at the interface. The scattering of waves at the interface is shown in Figure 2.

When the media of layer J and layer $(J+1)$ are the same, the matrix S^J is reduced to

$$S^J = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{18}$$

If the layered liquid is bounded at the top, $z = Z^0 = 0$, by a pressure free surface, and at the bottom, $z = Z^N$, by a rigid plane, both the wave vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ degenerate into scalars (vectors with a single element). The scattering matrices S^0 and S^N also degenerate into matrices with a single element, which can be derived by taking appropriate limit values of density and sound speed for one adjacent layer, zero density and zero speed for the upper layer of the free surface, and infinite density and infinite wave speed for the lower layer of the rigid plane. The limit values are

$$\begin{aligned} \hat{d}^0 &= S^0 \hat{a}^0 + \hat{s}^0(\kappa, \omega), \quad S^0 = -1, \\ \hat{d}^N &= S^N \hat{a}^N + \hat{s}^N(\kappa, \omega), \quad S^N = 1. \end{aligned} \tag{19}$$

Hence $\hat{a}^0 = \hat{d}^{01}$, $\hat{d}^N = \hat{d}^{N(N-1)}$, $\hat{a}^0 = \hat{a}^{01}$ and $\hat{a}^N = \hat{a}^{N(N-1)}$. We assume no explosive source at the free or rigid surfaces; the source wave vectors $\hat{s}^0$ and $\hat{s}^N$ vanish.

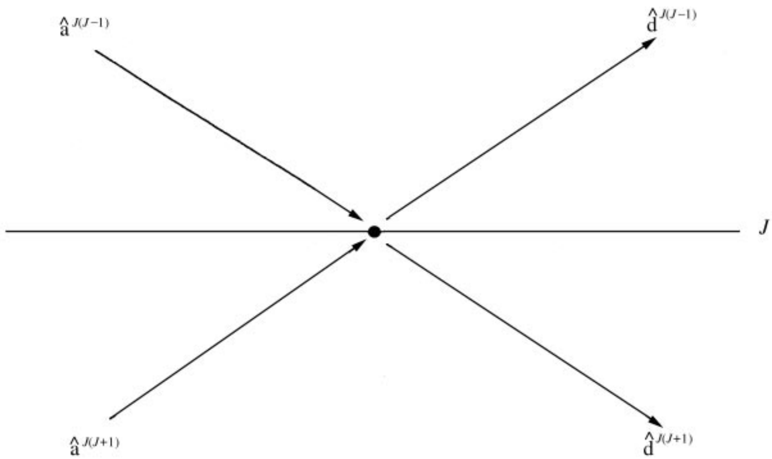

Figure 2. Scattering of waves at interface J.

If the bottom plane $z = Z^{N-1}$ is bounded by a semi-infinite liquid space, we let the plane $z = Z^N$ recede to infinity and the thickness $h^{N(N-1)} = h^{(N-1)N}$ approach infinity. From the radiation condition, the wave number in the semi-infinite space becomes complex, $\alpha^{(N-1)N} = \alpha^{N(N-1)} = [\kappa^2 - \omega^2/(c^{N(N-1)})^2]^{1/2}$, and the wave amplitude $\hat{d}^{(N-1)N}$ and $\hat{a}^{N(N-1)}$ must vanish. The elements of the 2×2 scattering matrix S^{N-1} should be modified accordingly.

3.2. GLOBAL SCATTERING MATRIX FOR THE MULTILAYERED MEDIUM

Combining $2N$ equations in equations (15) and (19), we can construct a system of equations for the entire multilayered medium in the following form:

$$\begin{pmatrix} \hat{d}^0 \\ \hat{\mathbf{d}}^1 \\ \hat{\mathbf{d}}^2 \\ \vdots \\ \hat{\mathbf{d}}^{N-1} \\ \hat{d}^N \end{pmatrix} = \begin{bmatrix} S^0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & S^1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & S^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S^{N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & S^N \end{bmatrix} \begin{pmatrix} \hat{a}^0 \\ \hat{\mathbf{a}}^1 \\ \hat{\mathbf{a}}^2 \\ \vdots \\ \hat{\mathbf{a}}^{N-1} \\ \hat{a}^N \end{pmatrix} + \begin{pmatrix} \hat{s}^0 \\ \hat{\mathbf{s}}^1 \\ \hat{\mathbf{s}}^2 \\ \vdots \\ \hat{\mathbf{s}}^{N-1} \\ \hat{s}^N \end{pmatrix}. \tag{20}$$

In compact notation, the previous equation is written as

$$\hat{\mathbf{d}} = \mathbf{S}\hat{\mathbf{a}} + \hat{\mathbf{s}}(\kappa, \omega). \tag{21}$$

The vector with $2N$ elements, $\hat{\mathbf{d}}$, which is named the global departing wave vector, represents waves departing from all interfaces downward and upward, and the vector $\hat{\mathbf{a}}$, which is named the global arriving wave vector, represents waves arriving at all interfaces upward and downward. The square matrix $\mathbf{S}$ which is a block-diagonal matrix of dimension $2N$ is called the global scattering matrix. The vector $\hat{\mathbf{s}}$ with $2N$ elements which is called the global source vector represents waves emitted from sources located at $r = 0$, $z = Z^J (J = 1, 2, \ldots, N-1)$.

Notice that two local co-ordinates $(r^{J(J-1)}, z^{J(J-1)})$ and $(r^{J(J+1)}, z^{J(J+1)})$ are used to analyze waves arriving and departing from the same interface; the amplitude coefficients are treated separately from the phase functions. In this section, the elements of S^J represent the reflection or transmission coefficients for waves incident at the interface J are the same as those calculated from a single co-ordinate. Since both vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are unknown quantities, we need an additional equation relating $\hat{\mathbf{d}}$ to $\hat{\mathbf{a}}$.

4. REVERBERATION MATRIX

The additional equation is supplemented by first noting that a wave departing from one side of the layer becomes the wave arriving at another side of the same layer. The amplitudes for the waves at both sides, however, differ by a phase shift factor as follows:

$$\hat{a}^{J(J-1)} = \mathrm{e}^{\mathrm{i}\alpha_j h_j} \hat{d}^{J(J-1)J},$$

$$\hat{d}^{J(J-1)} = \mathrm{e}^{-\mathrm{i}\alpha_j h_j} \hat{a}^{(J-1)J}, \quad j = 1, 2, \ldots, N. \tag{22}$$

We introduce a new local vector at the Jth interface, $\hat{\mathbf{d}}^{*j}$, and a new global vector $\hat{\mathbf{d}}^{*}$ for the departing waves as

$$\hat{\mathbf{d}}^{*j} = (\hat{d}^{(J-1)J}, \hat{d}^{(J+1)J})^{\mathrm{T}}, \quad \hat{\mathbf{d}}^{*} = (\hat{d}^{*0}, \hat{\mathbf{d}}^{*1}, \hat{\mathbf{d}}^{*2}, \ldots, \hat{\mathbf{d}}^{*(N-1)}, \hat{d}^{*N})^{\mathrm{T}}. \tag{23}$$

Here $\hat{d}^{*0} = \hat{d}^{10}$ and $\hat{d}^{*N} = \hat{d}^{(N-1)N}$. The global vectors $\hat{\mathbf{d}}^{*}$ and $\hat{\mathbf{d}}$ contain the same elements but are sequenced in different vectors. We may express this equivalence through a permutation matrix $\mathbf{U}$,

$$\hat{\mathbf{d}}^{*} = \mathbf{U}\hat{\mathbf{d}} \tag{24}$$

where $\mathbf{U}$ is a $2N \times 2N$ block-diagonal matrix composed of N same 2×2 sub-matrix U and other vanishing elements as

$$\mathbf{U} = \begin{bmatrix} U & 0 & \cdots & 0 \\ 0 & U & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U \end{bmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, all elements of wave vector $\hat{\mathbf{a}}$ are related to those of the vector $\hat{\mathbf{d}}^{*}$ as

$$\hat{\mathbf{a}} = \mathbf{P}(\kappa, \mathbf{h}, \omega)\hat{\mathbf{d}}^{*}, \tag{25}$$

where the total phase shift matrix $\mathbf{P}(\kappa, \mathbf{h}, \omega)$ or $\mathbf{P}(\mathrm{h})$ $(2N \times 2N)$ is a block-diagonal matrix, which is given by

$$\mathbf{P}(\mathbf{h}) = \begin{bmatrix} P(h^1) & 0 & \cdots & 0 \\ 0 & P(h^2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P(h^N) \end{bmatrix}, \quad P(h^j) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\alpha h)^J} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\alpha h)^J} \end{pmatrix}. \tag{26}$$

Substituting equation (24) into equation (25), we find the second equation that relates the vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ to be

$$\hat{\mathbf{a}} = \mathbf{P}(\mathbf{h})\mathbf{U}\hat{\mathbf{d}}. \tag{27}$$

Solving equations (21) and (27) simultaneously, we finally obtain

$$\hat{\mathbf{d}} = [\mathbf{I} - \mathbf{R}]^{-1}\hat{\mathbf{s}}(\kappa, \omega), \tag{28}$$

$$\hat{\mathbf{a}} = \mathbf{P}(\mathbf{h})\mathbf{U}[\mathbf{I} - \mathbf{R}]^{-1}\hat{\mathbf{s}}(\kappa, \omega), \tag{29}$$

where we have introduced the reverberation matrix $\mathbf{R}$ defined by

$$\mathbf{R}(\kappa, \omega) = \mathbf{S}\mathbf{P}(\mathbf{h})\mathbf{U}. \tag{30}$$

The matrix $[\mathbf{I} - \mathbf{R}(\kappa, \omega)]^{-1}$ relates the response of the multilayered medium to the excitation $\hat{\mathbf{s}}(\kappa, \omega)$ in the frequency-wavenumber domain. The dispersion relation for the resonant waves in the multilayered medium is given by

$$\det[\mathbf{I} - \mathbf{R}(\kappa, \omega)] = 0. \tag{31}$$

The determinant in equation (31) which is based on evaluation of scattering waves at each interface is in a form different from that derived by the Thomson and Haskell method (1950) which is based on the evaluation of the wave propagating from one interface to another. They should, however, yield the same numerical results for the dispersion relations in the $\omega - \kappa$ plane.

The frequency response for monochromatic waves in the layered medium is determined by completing inverse Hankel transform in equation (6), after substituting $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ into equation (8). The transient response of the same medium is determined by completing the inverse Fourier transform.

5. TRANSIENT WAVES IN THE MULTILAYERED LIQUID

Once the coefficient vectors $\hat{\mathbf{d}}$ and $\hat{\mathbf{a}}$ are known from equations (28) and (29), the complete list of potentials in the frequency domain will be expressed as

$$\hat{\boldsymbol{\Phi}}(\kappa, \mathbf{z}, \omega) = [\mathbf{P}(\mathbf{h} - \mathbf{z})\mathbf{U} + \mathbf{P}(\mathbf{z})][\mathbf{I} - \mathbf{R}(\kappa, \omega)]^{-1}\hat{\mathbf{s}}(\kappa, \omega), \tag{32}$$

where

$$\begin{aligned}\hat{\boldsymbol{\Phi}} = {} & \hat{\phi}^{01}(\kappa, z^{01}, \omega), \hat{\phi}^{10}(\kappa, z^{10}, \omega), \ldots, \hat{\phi}^{(N-1)N}(\kappa, z^{(N-1)N}, \omega), \hat{\phi}^{N(N-1)} \\ & (\kappa, z^{N(N-1)}, \omega)\}^{\mathrm{T}}\end{aligned} \tag{33}$$

and vertical co-ordinate vector of receivers

$$\mathbf{z} = \{z^{01}, z^{10}, \; z^{12}, z^{21}, \ldots, z^{(N-1)N}, z^{N(N-1)}\}^{\mathrm{T}}. \tag{34}$$

Applying inverse Hankel transform and Fourier transform, we can thus obtain the transient responses at N receives to the sources, i.e.,

$$\boldsymbol{\Phi}(\mathbf{r}, \mathbf{z}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\boldsymbol{\Phi}}(\mathbf{r}, \mathbf{z}, \omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{35}$$

where

$$\bar{\boldsymbol{\Phi}}(\mathbf{r}, \mathbf{z}, \omega) = \int_{0}^{\infty} \hat{\boldsymbol{\Phi}}(\kappa, \mathbf{z}, \omega)\,\mathrm{J}_0(\kappa\mathbf{r})\,\kappa\,\mathrm{d}\kappa \tag{36}$$

and

$$\mathbf{J}_0(\kappa\mathbf{r}) = \begin{bmatrix} J_0(\kappa r^{01}) & 0 & \cdots & 0 & 0 \\ 0 & J_0(\kappa r^{10}) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_0(\kappa r^{(N-1)N}) & 0 \\ 0 & 0 & \cdots & 0 & J_0(\kappa r^{N(N-1)}) \end{bmatrix}.$$

It is assumed that no two receivers are located in the same layer. We rewrite equation (35) in detail as

$$\boldsymbol{\Phi}(\mathbf{r}, \mathbf{z}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \int_{0}^{\infty} [\mathbf{P}(\mathbf{h} - \mathbf{z})\mathbf{U} + \mathbf{P}(\mathbf{z})][\mathbf{I} - \mathbf{R}(\kappa, \omega)]^{-1}\hat{\mathbf{s}}(\kappa, \omega)\mathbf{J}_0(\kappa\mathbf{r})\kappa\,\mathrm{d}\kappa. \tag{37}$$

For each element of the wave potentials in equation (37), the double-integral representation of the wave potentials, $\phi(r, z, t)$, can be calculated by either the traditional spectra method or the ray method, both methods being first proposed by Pekeries (1949). The ray method is particularly suitable if we expand the transfer function in a Neumann series

$$[\mathbf{I}-\mathbf{R}]^{-1}=\mathbf{I}+\mathbf{R}+\mathbf{R}^2+\cdots+\mathbf{R}^M+[\mathbf{I}-\mathbf{R}]^{-1}\mathbf{R}^{M+1}. \tag{38}$$

Substituting equation (38) into equation (37),

$$\boldsymbol{\Phi}(\mathbf{r},\mathbf{z},t)=\sum_{m=0}^{M}\boldsymbol{\Phi}^{(m)}(\mathbf{r},\mathbf{z},t)+\boldsymbol{\Phi}_R^{(M+1)}(\mathbf{r},\mathbf{z},\mathbf{t}), \tag{39}$$

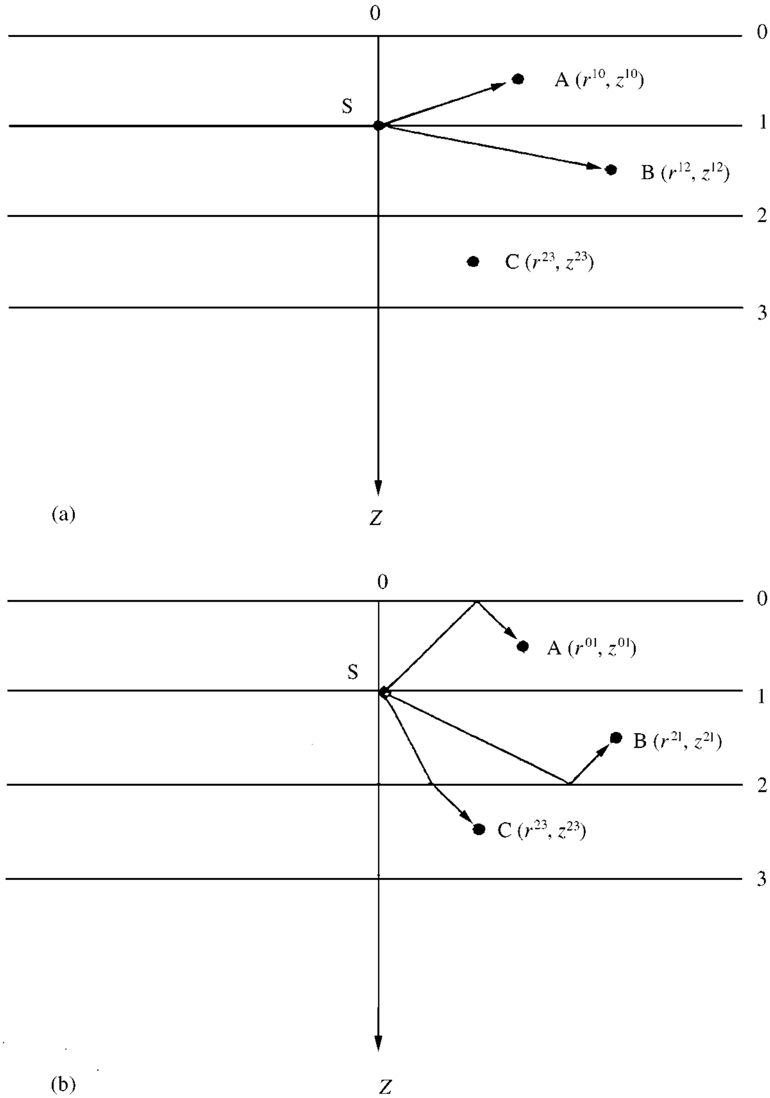

Figure 3. Rays with reverberations: (a) $\mathbf{P}(\mathbf{z})\mathbf{R}^0\mathbf{s}$; (b) $\mathbf{P}(\mathbf{z})\mathbf{R}\mathbf{s}$; (c) $\mathbf{P}(\mathbf{z})\mathbf{R}^2\mathbf{s}$.

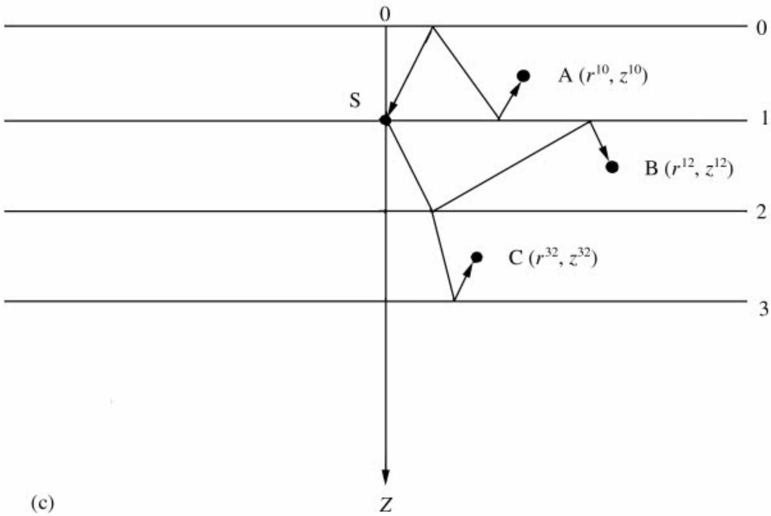

Figure 3. Continued

where

$$\boldsymbol{\Phi}^{(m)}(\mathbf{r}, \mathbf{z}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \int_{0}^{\infty} [\mathbf{P}(\mathbf{h}-\mathbf{z})\mathbf{U} + \mathbf{P}(\mathbf{z})]\mathbf{R}^{m}\hat{\mathbf{s}}(\kappa, \omega)\mathbf{J}_0(\kappa\mathbf{r})\,\kappa\,\mathrm{d}\kappa \quad (40)$$

and

$$\boldsymbol{\Phi}_R^{(M+1)}(\mathbf{r}, \mathbf{z}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \int_{0}^{\infty} [\mathbf{P}(\mathbf{h}-\mathbf{z})\mathbf{U} + \mathbf{P}(\mathbf{z})][\mathbf{I}-\mathbf{R}]^{-1}\mathbf{R}^{M+1} \times \hat{\mathbf{s}}(\kappa, \omega)\mathbf{J}_0(\kappa\mathbf{r})\,\kappa\,\mathrm{d}\kappa. \quad (41)$$

The integrals can thus be evaluated term by term to obtain the "generalized ray solution". The term $\boldsymbol{\Phi}^{(0)}(\mathbf{r}, \mathbf{z}, t)$ containing the factor $\mathbf{R}^0$ represents the waves originally generated by the applied forces, which propagate away from the sources to the receivers at $(\mathbf{r}, \mathbf{z})$. The term $\boldsymbol{\Phi}^{(1)}(\mathbf{r}, \mathbf{z}, t)$ containing the factor $\mathbf{R}$ represents the first set of reflections and transmissions of the direct waves in the multilayered liquid. In general, term $\boldsymbol{\Phi}^{(m)}(\mathbf{r}, \mathbf{z}, t)$ containing the factor $\mathbf{R}^m$ represents the set of m times reflections and transmissions of the source waves in the multilayered liquid. As an example, the rays arriving at the receivers for $\mathbf{R}^0$, $\mathbf{R}$, $\mathbf{R}^2$ in the three-layered liquid and a single point source at first interface are shown in Figure 3. The double-ray integrals can be calculated by applying the Cagniard method [11–13].

$\boldsymbol{\Phi}(\mathbf{r}, \mathbf{z}, t)$ in equation (39) can be evaluated by a recently developed hybrid method which combines the normal-mode and ray methods. In general, the former integrals $\mathbf{\Phi}^{(m)}(\mathbf{r}, \mathbf{z}, t)$ are generalized rays as formula (40), and the last integral, the inclusion of the remainder, may be obtained by normal-mode and the steepest descent method.

6. NUMERICAL EXAMPLE

For illustration, an example of a three-layer model of the shallow water as shown in Figure 4 is calculated. The point source with intensity of $\delta(r-0)\delta(z-13{\cdot}3)\times 1{\cdot}E5/2\pi r$ is towed at a depth of 13·3 km, and three receivers A, B and C, 15·5 km horizontally from the point source, are fixed at a depth of 10, 32·6 and 65·6 km, respectively. The three-layer model is divided into a four-layer model in order to

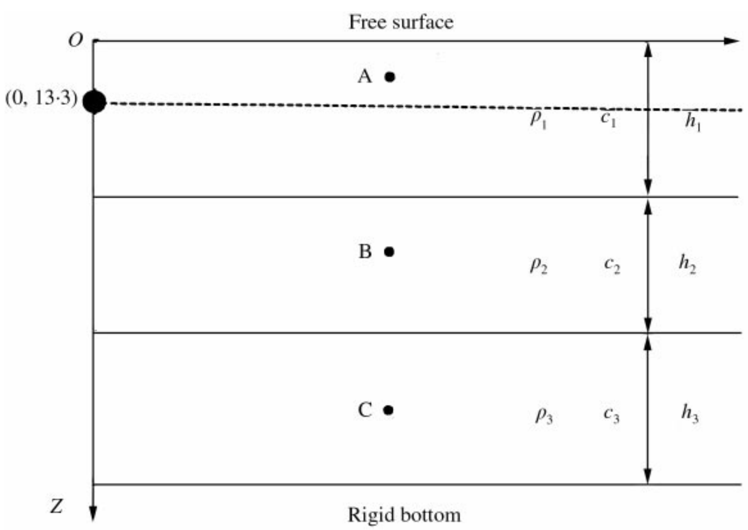

Figure 4. Three-layered model of shallow water with $h_1 = 22{\cdot}6$ km, $h_2 = 20{\cdot}3$ km, $h_3 = 22{\cdot}6$ km; $c_1 = 1500$ m/s, $c_2 = 1{\cdot}12c_1$, $c_3 = 1{\cdot}24c_1$; $\rho_1 = 1000$ kg/m^3, $\rho_2 = \rho_3 = 2\rho_2$. The source was towed at a depth of 13·3 km and three receivers locate at A(15·5, 10), B(15·5, 32·6) and C(15·5, 65·5).

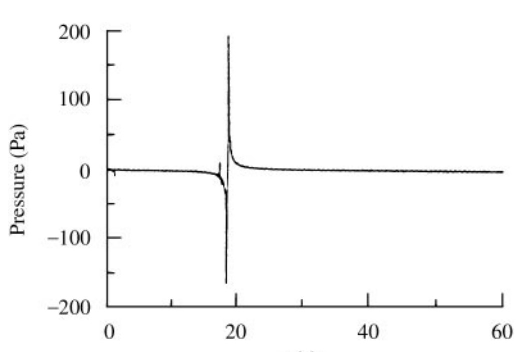

Figure 5. Response caused by the ray with respect to $\mathbf{R}$ at the receiver A.

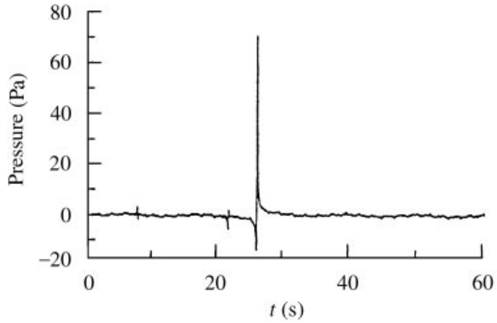

Figure 6. Response caused by the ray with respect to $\mathbf{R}^3$ at the receiver A.

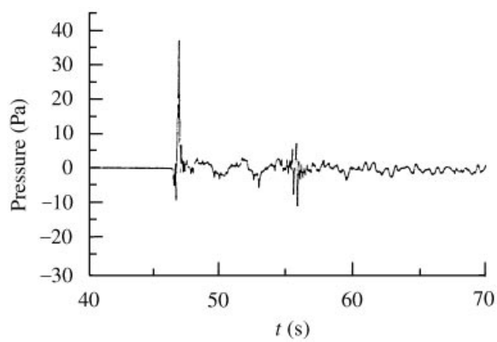

Figure 7. Response caused by the ray with respect to $\mathbf{R}^5$ at the receiver A.

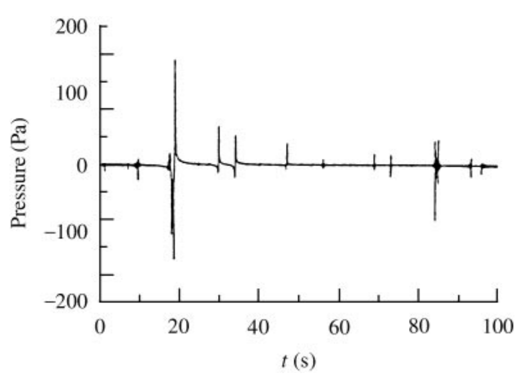

Figure 8. Response caused by the ray with respect to $\sum_{m=0}^{5}\mathbf{R}^m$ at the receiver A.

put the source point at the interface. The acoustical pressure at the receivers with respect to time is taken into account in the example. The responses with respect to $\mathbf{R}$, $\mathbf{R}^3$, $\mathbf{R}^5$ at the receiver A are shown in Figures 5–7, respectively, and that with regard to $\sum_{m=0}^{5}\mathbf{R}^m$ at the receiver A is given in Figure 8. The responses with respect to $\sum_{m=0}^{1}\mathbf{R}^m$, $\sum_{m=0}^{2}\mathbf{R}^m$ at receivers B and C are shown in Figures 9–12 respectively.

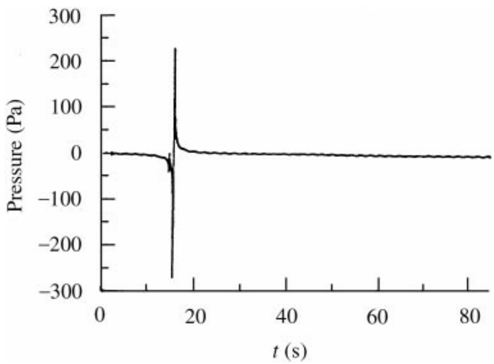

Figure 9. Response caused by the ray with respect to $\sum_{m=0}^{1}\mathbf{R}^m$ at the receiver B.

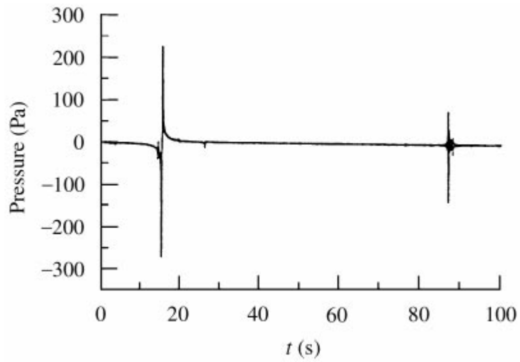

Figure 10. Response caused by the ray with respect to $\sum_{m=0}^{2}\mathbf{R}^m$ at the receiver B.

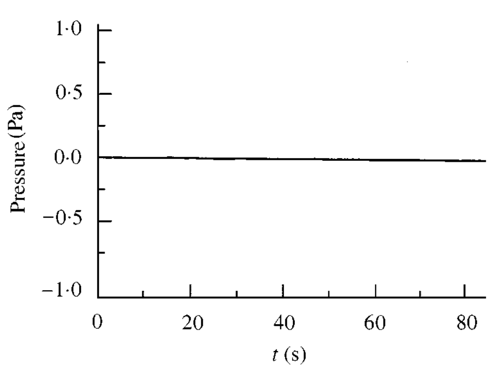

Figure 11. Response caused by the ray with respect to $\sum_{m=0}^{1}\mathbf{R}^m$ at the receiver C.

In Figure 5 and 6, the wave reflected one time from the surface and the waves reflected or transmitted three times from the surface and interfaces arrive at the receiver A at 18·47 and 29·67 s, respectively, which are the same as those obtained by the ray theory. In Figures 8–12, the summation of some rays are shown, and it is

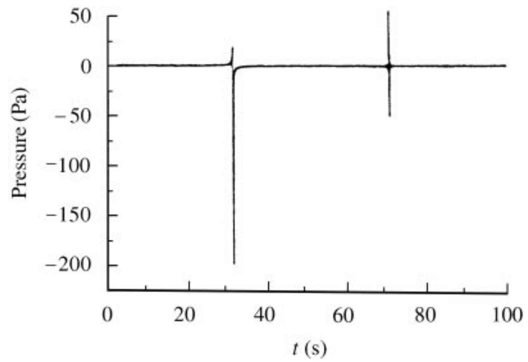

Figure 12. Response caused by the ray with respect to $\sum_{m=0}^{2}\mathbf{R}^m$ at the receiver C.

not difficult to find the arrival time of the reflected or transmitted waves. In particular, Figure 11 shows no response at receiver C from waves which suffer no scattering or scattered only once.

7. DISCUSSION & CONCLUSION

This paper presents the method of reverberation matrix (**R** matrix) for analyzing transient waves in a multilayered medium generated by a column of point sources. The waves within two adjacent layers are specified in terms of a two-element vector $\hat{\mathbf{a}}^J$ for the waves arriving at the interface J, and another vector $\hat{\mathbf{d}}^J$ for two waves departing from the same interface. By introducing a set of local co-ordinate originating from the interface, the local scattering matrix $\mathbf{S}^J$ (reflection and transmission) relating $\hat{\mathbf{a}}^J$ to $\hat{\mathbf{d}}^J$ is then determined. A source vector $\hat{\mathbf{s}}$ is also calculated if the interface contains a point source. The local matrices of all layers are then stacked to form the global scattering matrix for the entire medium.

Since within each layer the arriving waves at one interface is related to the departing waves at another interface by a phase shifting factor, the reverberation matrix **R** is then formulated by multiplying the global scattering matrix with the phase shift matrix and another permutation matrix as shown in equation (30). The **R** matrix characterizes the multiple reflections and transmissions of all waves in each and every layer, and the dispersion relation for waves in the medium is given by $\det(\mathbf{I}-\mathbf{R})=0$ (equation (31)). The Hankel–Fourier transformed response at any receiver can then be calculated from the values of $\hat{\mathbf{a}}$ or $\hat{\mathbf{d}}$, as shown in equations (28, 29). The inverse transform is accomplished without evaluating the residues by expanding $(\mathbf{I}-\mathbf{R})^{-1}$ into a power series of **R**. The integral containing factor $\mathbf{R}^m$ represents the wave that is reflected by the interfaces or transmitted through the interfaces m times along a specific path. The double integrals of each term in series expansion can then be evaluated by applying the Cagniard method. When the method of propagator matrix is applied, a state vector which is composed of normal components of stress and velocity at each horizontal plane is introduced, and the state vector at the plane is related to that through the transfer matrix (**T** matrix), $\hat{\mathbf{u}}(z) = \mathbf{T}(z, z_j)\hat{\mathbf{u}}(z_J)$. For a multilayered medium with interface $J = 1, 2, 3, \ldots, N$, the state vector $\hat{\mathbf{u}}(z_N)$ is related to that at $z = z_0$ by the product of **T** matrices through the applications of continuity conditions at the interfaces. The unknown variables of the state vector at each plane are then determined by reformulating the matrix product to satisfy the boundary conditions at both ends.

In comparison with the method of reverberation matrix, the state vector can be calculated from the wave vector $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ within each layer, and the latter are determined from the scattering matrix for which the boundary conditions at both ends had already been satisfied. The dispersion relation (frequency equation) is thus given directly from $(\mathbf{I} - \mathbf{R})$ without reformulating the matrix. This, of course, is at the expense of increasing the size of matrices. Furthermore, the **R** matrix formulation is particularly convenient for analyzing the transient or steady state waves along the specific ray paths, and the arrival time of a wave through the path can readily be calculated.

Only a column of point sources, one in each layer, is considered in this article. Additional point sources located in the same column can be treated by introducing fictitious interface through the source at the expense of increasing the number of layers. A number of sources located arbitrarily in the medium can also be treated if the three-dimensional Cartesian co-ordinate system is adopted to reformulate the local scattering matrix. The method can also be extended to analyze the reverberation of elastic waves (**P** and **S**-wave) in layered solid media.

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